Localization of the Spectra of P- and P_0-Matrices

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ABSTRACT

It is well known that if $S=\{\lambda_1,\dots,\lambda_n\}$ is a spectrum of a P-matrix, then $|\arg\lambda_i|<\pi-\pi/n$. We show that if the number of elements of S in the right half plane, or in the left half plane, is given, then the above bound can be improved, namely, there exists α such that $|\arg\lambda_i|<\alpha<\pi-\pi/n$. When S has exactly one element in the right half plane, it is shown that $|\arg\lambda_i|<\frac{2}{3}\pi$, independently of n.

1. INTRODUCTION

A *P-matrix* [*P_0-matrix*] [3] is a matrix all of whose principal minors are positive [nonnegative]. Such matrices are related to stable matrices (e.g. [5]) and play an important role in economics and mathematical programming (e.g. [1]). They include the important class of nonsingular M-matrices introduced by Ostrowski [8].

Let $S=\{\lambda_1,\dots,\lambda_n\}$, and let

$$\sigma_k(S)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\prod_{t=1}^{k}\lambda_{i_t},\qquad k=1,\dots,n,$$

denote the kth elementary symmetric function of the numbers $\lambda_1,\dots,\lambda_n$.

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If S is the spectrum of a matrix having positive [nonnegative] sums of principal minors, then it satisfies

$$\sigma_k(S) > 0, \qquad k = 1, \dots, n \tag{1}$$

$$\left[\sigma_k(S) \geqslant 0, \qquad k = 1, \dots, n.\right] \tag{1_0}$$

In particular, the spectrum of a P-matrix [P_0-matrix] satisfies (1) [(1_0)]. In fact, it is shown in [6] that if S satisfies (1) [(1_0)], then there exists a P-matrix [P_0-matrix] whose spectrum is S. Thus S satisfies (1) [(1_0)] if and only if it is the spectrum of some P-matrix [P_0-matrix]. Such a set will be called a *P-set* [*P_0-set*].

Observe that a P-set [P_0-set] consists of positive [nonnegative] numbers (see for example Theorem 1 below) and of conjugate pairs. Observe also that S is a P_0-set if and only if $S \cup \{0\}$ is a P_0-set. Thus we consider in this paper P_0-sets which do not contain zero elements.

Kellogg [7] proved that elements of a P-set cannot lie in a given wedge around the negative axis. More precisely he proved

THEOREM 1 (Kellogg).

(a) *If $\{\lambda_1, \dots, \lambda_n\}$ is a P-set, then*

$$|\arg \lambda_i| < \pi - \frac{\pi}{n}, \qquad i = 1, \dots, n. \tag{2}$$

(b) *If $S = \{\lambda_1, \dots, \lambda_n\}$, $\lambda_i \neq 0$, $i = 1, \dots, n$, is a P_0-set, then*

$$|\arg \lambda_i| \leqslant \pi - \frac{\pi}{n}, \qquad i = 1, \dots, n. \tag{2_0}$$

Equality in (2_0) holds if and only if

$$\left.\begin{aligned} &\sigma_k(S) = 0, \qquad k = 1, \dots, n-1, \\ &\sigma_n(S) > 0. \end{aligned}\right. \tag{3}$$

For a set S, we denote by $\pi(S)$ and $\nu(S)$ the number of elements of S which have positive and negative real parts respectively.

In this paper we further investigate the localization of the elements of P-sets.

A natural question is whether for a P-set or a P_0-set $S=\{\lambda_1,\dots,\lambda_n\}$ such that $\pi(S)$ or $\nu(S)$ is given, the region described by (2) can be reduced, i.e., whether there exists a number α satisfying

$$|\arg\lambda_i|<\alpha<\pi-\frac{\pi}{n},\qquad i=1,\dots,n. \tag{4}$$

Clearly $\pi(S)\geqslant 1$ when the cardinality of S, denoted by $|S|$, is odd, and $\pi(S)\geqslant 2$ when $|S|$ is even. It is also clear (e.g. [6]) that if n is odd [even], then there exists a P-set S such that $|S|=n$ and $\pi(S)=1$ [$\pi(S)=2$].

In Section 2 it is shown that when $\pi(S)=1$ (and $n>3$), $\alpha=\frac{2}{3}\pi$ satisfies (4). This value is independent of n and cannot be improved. An analogous result is obtained for P_0-sets.

In Section 3 we show that there exists α satisfying (4) when $\pi(S)=2$ or $\nu(S)=2$ $(n>6)$. We conjecture that when $\pi(S)=2$, (4) holds with $\alpha=\frac{5}{6}\pi$ independently of n.

Similar improvements of the bound in (2) are given in Section 4 when $\pi(S)=k$ or $\nu(S)=k$.

2. SETS S WITH $\pi(S)=1$

We start with two lemmas.

LEMMA 1. *Let S consist of real numbers and conjugate pairs such that $\nu(S)=0$. Then S is a P_0-set.*

Proof. We show, by induction on $|S|$, that S satisfies (1_0). The claim is immediate for sets of one or two elements. Assume it is valid for every set $S_0=\{\lambda_1,\dots,\lambda_n\}$ of n elements, and consider the sets $S_1=S_0\cup\{a\}$ and $S_2=S_0\cup\{a+bi,a-bi\}$, where $a\geqslant 0$. Then

$$\sigma_1(S_1)=a+\sigma_1(S_0)\geqslant 0,$$

$$\sigma_k(S_1)=a\sigma_{k-1}(S_0)+\sigma_k(S_0)\geqslant 0,\qquad k=2,\dots,n,$$

$$\sigma_{n+1}(S_1)=a\sigma_n(S_0)\geqslant 0$$

and

$$\sigma_1(S_2) = 2a + \sigma_1(S_0) \geqslant 0$$

$$\sigma_2(S_2) = a^2 + b^2 + 2a\sigma_1(S_0) + \sigma_2(S_0) \geqslant 0$$

$$\sigma_k(S_2) = (a^2 + b^2)\sigma_{k-2}(S_0) + 2a\sigma_{k-1}(S_0) + \sigma_k(S_0) \geqslant 0, \qquad k = 3,\dots,n$$

$$\sigma_{n+1}(S_2) = (a^2 + b^2)\sigma_{n-1}(S_0) + 2a\sigma_n(S_0) \geqslant 0$$

$$\sigma_{n+2}(S_2) = (a^2 + b^2)\sigma_n(S_0) \geqslant 0 \qquad \blacksquare$$

LEMMA 2. *Let S consist of real numbers and conjugate pairs such that* $\pi(S) = 0$. *Then*

$$\sigma_k(S) \geqslant 0 \qquad \textit{if} \quad k \textit{ is even}$$

and

$$\sigma_k(S) \leqslant 0 \qquad \textit{if} \quad k \textit{ is odd}.$$

Proof. Let $S = \{\lambda_1,\dots,\lambda_n\}$. The set $T = \{-\lambda_1,\dots,-\lambda_n\}$ satisfies the conditions of Lemma 1 and

$$\sigma_k(S) = (-1)^k \sigma_k(T). \qquad \blacksquare$$

THEOREM 2. *Let* $S = \{\lambda_1,\dots,\lambda_n\}$ *be a P-set such that* $\pi(S) = 1$. *Then*

$$|\arg \lambda_i| < \tfrac{2}{3}\pi, \qquad i = 1,\dots,n. \tag{5}$$

Proof. Without loss of generality we can assume that $\lambda_1 > 0$ and that $\lambda_2,\dots,\lambda_n$ have nonpositive real parts. Let $\lambda \in S$, $\lambda \neq \lambda_1$. Clearly $\bar{\lambda} \in S$. Let

$$A_k = \sigma_k(S), \qquad k = 1,\dots,n,$$

$$B_k = \sigma_k(\{\lambda_1, \lambda, \bar{\lambda}\}), \qquad k = 1,2,3$$

and

$$C_k = \sigma_k(S \setminus \{\lambda_1, \lambda, \bar{\lambda}\}), \qquad k = 1,\dots,n-3.$$

By Lemma 2,

$$\begin{aligned} C_k &\geq 0 \qquad \text{if} \quad k \text{ is even,} \\ C_k &\leq 0 \qquad \text{if} \quad k \text{ is odd.} \end{aligned} \tag{6}$$

Clearly,

$$B_1 + C_1 = A_1 > 0,$$

$$B_3 C_{n-3} = A_n > 0.$$

Since $\pi(S) = 1$, n is odd and it follows from (6) that

$$B_1 > 0, \qquad B_3 > 0. \tag{7}$$

Also

$$B_2 C_{n-3} + B_3 C_{n-4} = A_{n-1} > 0,$$

so by (6) and (7)

$$B_2 > 0. \tag{8}$$

The inequality (5) follows from Theorem 1, (7), and (8). ∎

It is easy to check that the set $\{1+\varepsilon, e^{\frac{2}{3}\pi i}+\varepsilon, e^{-\frac{2}{3}\pi i}+\varepsilon\}$, $\varepsilon > 0$, is a P-set. As shown in [6] for any P-set S, there exists a number λ such that $S_1 = S \cup \{\lambda, \bar{\lambda}\}$ is also a P-set and $\pi(S_1) = \pi(S)$. Thus the inequality (5) cannot be improved.

For P_0-sets we have the following.

THEOREM 3. *Let* $S = \{\lambda_1, \dots, \lambda_n\}$, $\lambda_i \neq 0$, $i = 1, \dots, n$, *be a* P_0*-set such that* $\pi(S) = 1$. *Then*

$$|\arg \lambda_i| \leq \tfrac{2}{3}\pi, \qquad i = 1, \dots, n. \tag{9}$$

Equality in (9) *holds for* $i = j$ *only if*

$$S = \{|\lambda_j|, \lambda_j, \bar{\lambda}_j\} \cup S_1$$

where S_1 *consists of pairs of conjugate pure imaginary numbers.*

Proof. We follow the proof of Theorem 2 with $\lambda = \lambda_j$. Since now we deal with a P_0-set, A_1 and A_{n-1} will be nonnegative (instead of positive). Thus we get

$$B_1 \geqslant 0, \qquad B_2 \geqslant 0, \qquad B_3 > 0.$$

By Theorem 1, equality in (9), for $i = j$, implies that $B_1 = B_2 = 0$. Therefore $\lambda_1 = |\lambda_j|$. By (6), $C_1 \leqslant 0$. As $B_1 = 0$, $C_1 = A_1 \geqslant 0$, so

$$\sigma_1(S_1) = C_1 = 0. \tag{10}$$

Since $\pi(S_1) = 0$, it follows that $\nu(S_1) = 0$. ∎

We end this section with a consequence of Theorem 2 which relates to a conjecture of Carlson on weakly sign-symmetric matrices.

An $n \times n$ P-matrix A is called *weakly sign-symmetric* [2] if

$$A(\alpha, \beta) A(\beta, \alpha) \geqslant 0 \tag{11}$$

for all $\alpha, \beta \subseteq \{1, \dots, n\}$ such that $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$, where $A(\alpha, \beta)$ is the minor whose rows are indexed by α and whose columns are indexed by β.

Carlson [2] conjectured that if S is the spectrum of an $n \times n$ weakly sign-symmetric matrix A, then $\pi(S) = n$, i.e., A is stable. The following proposition may be of some interest in studying the conjecture.

THEOREM 4. *Let A be an $n \times n$ weakly sign-symmetric matrix, where n is odd, and let $S = \{\lambda_1, \dots, \lambda_n\}$ be the spectrum of A. Then $\pi(S) \geqslant 3$.*

Proof. Suppose the claim is not true. Since n is odd, $\pi(S) = 1$. Without loss of generality we can assume that $\lambda_1 > 0$. By Theorem 2,

$$\tfrac{1}{2}\pi \leqslant |\arg \lambda_i| \leqslant \tfrac{2}{3}\pi, \qquad i = 2, \dots, n. \tag{12}$$

Let $T = \{\mu_1, \dots, \mu_n\}$, $\mu_i = \lambda_i^2$, be the spectrum of $B = A^2$. By (12)

$$\tfrac{2}{3}\pi < |\arg \mu_i| \leqslant \pi. \tag{13}$$

Using the Cauchy-Binet formula (e.g. [4, p. 9]) and the fact that A is weakly sign-symmetric, we get that all the principal minors of B of orders 1, $n-1$, and n are positive. Let $M_k = \sigma_k(T)$. Then

$$M_1 > 0, \qquad M_{n-1} > 0, \qquad M_n > 0.$$

Consider the elementary symmetric functions of $\{\mu_1, \mu_i, \bar{\mu}_i\}$, $i > 1$. By a computation similar to that of Theorem 2 we get that

$$|\arg \mu_i| < \tfrac{2}{3}\pi,$$

contradicting (13). ∎

The analogous conjecture and result for P_0-matrices are false. The P_0-matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

satisfies (11) but has only one eigenvalue in the right half plane. Notice that the spectrum of $A + \varepsilon I$ is the set introduced after Theorem 2. This does not contradict Theorem 4, as $A + \varepsilon I$ does not satisfy (11) for $\varepsilon > 0$.

3. SETS S WITH $\pi(S) = 2$ OR $\nu(S) = 2$

In the previous section we studied the case $\pi(S) = 1$. Now we consider P_0-sets for which $\pi(S) = 2$ (so $|S|$ is even). Similar discussion is carried for $\nu(S) = 2$. [Obviously $\nu(S) \neq 1$ for P_0-sets.]

The following lemmas are based on continuity arguments.

LEMMA 3. *There exists $\delta > 0$ such that if*

$$|a_i| \leqslant \delta, \qquad i = 1, \dots, n,$$

then all the roots of the polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

lie inside the unit circle.

Proof. For δ small enough $p(z)$ is close to the polynomial z^n, whose roots are zero. ∎

LEMMA 4. *Let $d > 0$ and $n \geqslant 7$. There exists $\varepsilon(d)$, $d > \varepsilon(d) > 0$, such that if*

$$|a_i| \leqslant \varepsilon(d), \qquad i = 1,\dots,n-1,$$

then the set S of the roots of

$$q(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + d$$

satisfies

$$\pi(S) > 2 \quad \textit{and} \quad \nu(S) > 2. \tag{14}$$

Proof. For $\varepsilon(d)$ small enough, $q(z)$ is close to $z^n + d$. ■

LEMMA 5. *Let $d > 0$ and $n \geqslant 7$. If*

$$|a_i| \leqslant \varepsilon(d), \qquad i = 1,\dots,n-1,$$

and

$$a_n > d,$$

then the set $S = \{z_1,\dots,z_n\}$ of the roots of

$$r(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$$

satisfies (14).

Proof. Let $c = \sqrt[n]{a_n/d}$. Consider the set $T = \{z_1/c,\dots,z_n/c\}$. T is the set of the roots of

$$t(z) = z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n,$$

where

$$b_k = \frac{a_k}{c^k}, \qquad k = 1,\dots,n.$$

Since $c > 1$,

$$|b_k| = \frac{|a_k|}{c^k} < |a_k| \leqslant \varepsilon(d), \qquad k = 1,\dots,n-1,$$

and $b_n = a_n/c^n = d$.

By Lemma 4, (14) holds for T and thus for S. ∎

THEOREM 5. *Let* $S = \{\lambda_1,\dots,\lambda_n\}$, $\lambda_i \neq 0$, $i = 1,\dots,n$, $n > 6$, *be a* P_0*-set such that* $\pi(S) = 2$ *or* $\nu(S) = 2$. *Then there exists* α *satisfying*

$$|\arg \lambda_i| < \alpha < \pi - \frac{\pi}{n}, \qquad i = 1,\dots,n. \tag{4}$$

Proof. Let δ be the number whose existence was established in Lemma 3, and let $\varepsilon = \varepsilon(\delta)$. Denote

$$\beta = \min\left\{ \frac{\arcsin\left(\varepsilon \sin \dfrac{\pi}{n^2}\right)}{n}, \frac{\pi}{n^2} \right\}. \tag{15}$$

We show that $\alpha = \pi - \pi/n - \beta$ satisfies (4).

Suppose there exists $\lambda \in S$ such that

$$\pi - \frac{\pi}{n} \geqslant \arg \lambda \geqslant \alpha. \tag{16}$$

Without loss of generality we can assume that $|\lambda| = 1$. Let

$$A_k = \sigma_k(S), \qquad k = 1,\dots,n.$$

The set $T = \{-\lambda_1,\dots,-\lambda_n\}$ consists of the roots of

$$p(z) = z^n + A_1 z^{n-1} + \cdots + A_{n-1} z + A_n.$$

Let θ be the argument of $-\bar{\lambda}$. By (16),

$$\frac{\pi}{n} \leqslant \theta \leqslant \frac{\pi}{n} + \beta. \tag{17}$$

By (15) and (17)

$$\frac{\pi}{n} \leqslant k\theta \leqslant \pi - \frac{\pi}{n^2}, \qquad k = 1,\dots,n-1.$$

Thus

$$\sin k\theta \geqslant \sin \frac{\pi}{n^2} > 0, \qquad k = 1,\dots,n-1. \tag{18}$$

Also by (15) and (17)

$$\pi \leqslant n\theta \leqslant \pi + \arcsin\left(\varepsilon \sin \frac{\pi}{n^2}\right).$$

Thus

$$0 \leqslant -\sin n\theta \leqslant \varepsilon \sin \frac{\pi}{n^2}. \tag{19}$$

As $-\bar{\lambda} \in T$,

$$p(-\bar{\lambda}) = e^{in\theta} + \sum_{k=0}^{n-1} A_{n-k} e^{ik\theta} = 0. \tag{20}$$

The imaginary part of this equation implies that

$$\sum_{k=1}^{n-1} A_{n-k} \sin k\theta = -\sin n\theta.$$

Since $A_k \geqslant 0$, $k = 1,\dots,n$, it follows by (18) that

$$A_{n-k} \sin k\theta \leqslant -\sin n\theta, \qquad k = 1,\dots,n-1.$$

By (18) and (19),

$$A_{n-k} \sin \frac{\pi}{n^2} \leqslant \varepsilon \sin \frac{\pi}{n^2}, \qquad k = 1,\dots,n-1.$$

Thus,

$$A_k \leqslant \varepsilon, \qquad k = 1, \ldots, n-1.$$

By Lemma 3, $A_n > \delta$, since $|-\bar{\lambda}| = 1$. Thus $p(z)$ satisfies the conditions of Lemma 5, so $\pi(S) = \nu(T) > 2$ and $\nu(S) = \pi(T) > 2$. This contradicts the assumptions of the theorem, so there exists no $\lambda \in S$ satisfying (16). ■

Recall the condition for equality in Theorem 1. Sets which satisfy (3) will be called *extremal P_0-sets*. It is pointed out in [6] that if $\{\lambda_1, \ldots, \lambda_n\}$ is a P_0-set, then $\{\lambda_1 + \varepsilon, \ldots, \lambda_n + \varepsilon\}$, $\varepsilon > 0$, is a P-set. The extremal P_0-sets for $n \leqslant 6$ demonstrate, by this observation, that Theorem 5 is not true, neither for $n \leqslant 5$, nor for $n = 6$ and $\nu(S) = 2$, even for P-sets. To show that it is not true for $n = 6$ and $\pi(S) = 2$, let λ_1, λ_2, and λ_3 be numbers on the unit circle satisfying

$$\operatorname{Re}\lambda_1 = \tfrac{1}{2}\sqrt{3},$$

$$\operatorname{Re}\lambda_2 = -\tfrac{1}{2}\sqrt{3} + \varepsilon,$$

$$\operatorname{Re}\lambda_3 = -\tfrac{1}{2}\varepsilon,$$

and let $S = \{\lambda_1, \lambda_2, \lambda_3, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3\}$. For $0 < \varepsilon < \frac{1}{2}\sqrt{3}$, S is a P-set satisfying $\pi(S) = 2$. For ε small enough, $|\arg\lambda_2|$ is close to $\frac{5}{6}\pi$.

Recall the remark following Theorem 2 that if S is a P-set, then there exists λ, $\operatorname{Re}\lambda < 0$, such that $S \cup \{\lambda, \bar{\lambda}\}$ is also a P-set. Obviously for every $\mu > 0$, $S \cup \{\mu\}$ is a P-set. Thus the bound α in Theorem 5 cannot be smaller than $\frac{5}{6}\pi$. The largest n for which there exists an extremal P_0-set S such that $\pi(S) = 1$ is $n = 3$. The largest argument of an element in such a set is $\frac{2}{3}\pi$. In Theorem 2 we proved that this number is the bound α in (4) in the case $\pi(S) = 1$. The largest n for which there exists an extremal P_0-set S such that $\pi(S) = 2$ or $\nu(S) = 2$ is 6. The corresponding argument is $\frac{5}{6}\pi$. Motivated by this observation, we conjecture

Conjecture 1. Let $S = \{\lambda_1, \ldots, \lambda_n\}$ be a P-set such that $\pi(S) = 2$. Then

$$|\arg\lambda_i| < \tfrac{5}{6}\pi, \qquad i = 1, \ldots, n.$$

Conjecture 2. Let $S = \{\lambda_1, \ldots, \lambda_n\}$ be a P-set such that $\nu(S) = 2$. Then

$$|\arg\lambda_i| < \tfrac{5}{6}\pi, \qquad i = 1, \ldots, n.$$

The conjectures are stated separately because their confirmations may be independent. Similar conjectures may be stated for P_0-sets.

Let S be a P_0-set with $\pi(S)=2$. We conclude this section with a bound for the argument of the elements in the right half plane. This result may be of some use in studying the conjectures.

THEOREM 6. *Let* $S=\{\lambda_1,\dots,\lambda_n\}$, $n\geqslant 4$; $\lambda_i\neq 0$, $i=1,\dots,n$; $\operatorname{Re}\lambda_1>0$; $\operatorname{Re}\lambda_2>0$; $\operatorname{Re}\lambda_i\leqslant 0$, $i=3,\dots,n$; *be a* P_0*-set and let* $\theta=\max|\arg\lambda_i|$. *Then*

$$|\arg\lambda_1|=|\arg\lambda_2|\leqslant\pi-\theta. \tag{21}$$

Proof. Choose j, $3\leqslant j\leqslant n$. Denote by T the set $\{\lambda_1,\lambda_2,\lambda_j,\bar{\lambda}_j\}$. Let

$$A_k=\sigma_k(T),\qquad k=1,\dots,4,$$

$$B_k=\sigma_k(S\setminus T),\qquad k=1,\dots,n-4,$$

$$C_k=\sigma_k(S),\qquad k=1,\dots,n.$$

If $n=4$, then $T=S$ and $A_k\geqslant 0$, $k=1,\dots,4$.
Since $\pi(S)=2$, n is even, so if $n>4$, then $n\geqslant 6$. In this case

$$A_1+B_1=C_1\geqslant 0,$$

$$A_3B_{n-4}+A_4B_{n-5}=C_{n-1}\geqslant 0.$$

By Lemma 2,

$$B_1\leqslant 0,\qquad B_{n-5}\leqslant 0,\qquad B_{n-4}>0,$$

so

$$A_1\geqslant 0\quad\text{and}\quad A_3\geqslant 0.$$

If λ_1 and λ_2 are positive then (21) is obvious. Otherwise, let

$$\lambda_1=re^{i\varphi},\quad 0<\varphi<\tfrac{1}{2}\pi\qquad\left(\lambda_2=re^{-i\varphi}\right), \tag{22}$$

$$\lambda_j=\rho e^{i\theta_j},\quad \tfrac{1}{2}\pi\leqslant\theta_j<\pi \tag{23}$$

(there is no loss of generality in choosing λ_j in the upper half plane). Then

$$A_1 = 2r\cos\varphi + 2\varphi\cos\theta_j \geqslant 0,$$

$$A_3 = 2r\rho\left(\rho\cos\varphi + r\cos\theta_j\right) \geqslant 0.$$

Therefore

$$\cos\varphi \geqslant -\frac{\rho}{r}\cos\theta_j = \frac{\rho}{r}\cos\left(\pi - \theta_j\right),$$

$$\cos\varphi \geqslant \frac{r}{\rho}\cos\theta_j = \frac{r}{\rho}\cos\left(\pi - \theta_j\right)$$

Thus

$$\cos\varphi \geqslant \cos\left(\pi - \theta_j\right). \tag{24}$$

The inequality (21) follows from (22), (23), and (24), which holds for all j, $3 \leqslant j \leqslant n$. ■

We remark that if $S = \{\lambda_1, \dots, \lambda_n\}$ is a P-set, then so is $T = \{\lambda_1 - \varepsilon, \dots, \lambda_n - \varepsilon\}$ if $\varepsilon > 0$ is small enough. Thus if S is a P-set, (21) holds with strict inequality.

4. SETS WITH $\pi(S) = k$ OR $\nu(S) = k$

We consider now the general case. Let S be the set of the roots of $z^n + d$, $d > 0$. Observe that

$$\begin{aligned} n = 2l, \quad l \text{ even} \quad &\Rightarrow \quad \pi(S) = \nu(S) = l, \\ n = 2l, \quad l \text{ odd} \quad &\Rightarrow \quad \pi(S) = \nu(S) = l - 1, \\ n = 2l + 1, \quad l \text{ even} \quad &\Rightarrow \quad \pi(S) = l, \quad \nu(S) = l + 1, \\ n = 2l + 1, \quad l \text{ odd} \quad &\Rightarrow \quad \pi(S) = l + 1, \quad \nu(S) = l. \end{aligned} \tag{25}$$

Based on this observation, we generalize the results of the previous section.

In the sequel k and n denote natural numbers.

LEMMA 6. *Let $d>0$ and $n \geqslant 2k+3$. There exists $\varepsilon(d)$, $d>\varepsilon(d)>0$, such that if $|a_i| \leqslant \varepsilon(d)$, $i=1,\dots,n-1$, then the set S of the roots of $q(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+d$ satisfies*

$$\pi(S)>k \quad \text{and} \quad \nu(S)>k. \tag{26}$$

LEMMA 7. *Let $d>0$ and $n \geqslant 2k+3$. If $|a_i| \leqslant \varepsilon(d)$, $i=1,\dots,n-1$, and $a_n>d$, then the set S of the roots of $r(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ satisfies* (26).

THEOREM 7. *Let $S=\{\lambda_1,\dots,\lambda_n\}$, $\lambda_i \neq 0$, $i=1,\dots,n$, $n>2k+2$, be a P_0-set such that $\pi(S)=k$ or $\nu(S)=k$ (notice that $\nu(S)$ must be even). Then there exists α satisfying*

$$|\arg\lambda_i|<\alpha<\pi-\frac{\pi}{n}, \qquad i=1,\dots,n. \tag{4}$$

The proofs of Lemmas 6 and 7 are based on (25) and are essentially the same as those of Lemmas 4 and 5. The proof of Theorem 7 uses Lemmas 1, 6, and 7 and is similar to that of Theorem 5. The extremal P_0-sets of order $2k+1$, where k is odd, and $2k+2$, where k is even, demonstrate that the theorem does not hold for $n \leqslant 2k+2$.

In the proof of Theorem 6 it was shown that A_1 and A_3 are nonnegative. In the general case of a P_0-set S, without zero elements and with $\pi(S)=k$, let $\lambda_1,\dots,\lambda_k$ be the elements of S in the right half plane and let $\lambda \in S$ satisfy $\operatorname{Re}\lambda \leqslant 0$. Then

$$A_1=\sigma_1(\{\lambda_1,\dots,\lambda_k,\lambda,\bar{\lambda}\}) \geqslant 0$$

and

$$\tilde{A}_{k+1}=\sigma_{k+1}(\{\lambda_1,\dots,\lambda_k,\lambda,\bar{\lambda}\}) \geqslant 0.$$

By (25), the largest n for which there exists an extremal P_0-set S such that $\pi(S)=k$ or $\nu(S)=k$ is $2k+1$, where k is odd, or $2k+2$, where k is even.

The paper is concluded with the following generalizations of Conjectures 1 and 2.

CONJECTURE 3. Let $S=\{\lambda_1,\dots,\lambda_n\}$ be a P-set such that $\pi(S)=k$ where k is odd. Then

$$|\arg\lambda_i|<\frac{2k}{2k+1}\pi, \qquad i=1,\dots,n.$$

CONJECTURE 4. Let $S = \{\lambda_1, \dots, \lambda_n\}$ be a P-set such that $\pi(S) = k$ where k is even. Then

$$|\arg \lambda_i| < \frac{2k+1}{2k+2}\pi, \qquad i = 1, \dots, n.$$

CONJECTURE 5. Let $S = \{\lambda_1, \dots, \lambda_n\}$ be a P-set such that $\nu(S) = k$ (k must be even). Then

$$|\arg \lambda_i| < \frac{2k+1}{2k+2}\pi, \qquad i = 1, \dots, n.$$

Here too, similar conjectures may be stated for P_0-sets.

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